B.Sc. Part 2 (HONS) 4th Paper

Differential Equations

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Orthogonal Trajectories of POLAR CURVE

Method

1. Find $\frac{dr}{d\theta}$.
2. Replace $\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$.
3. Solve the equation. The result is the required orthogonal trajectory.

Examples 1 Find the orthogonal trajectories of $r = a(1+\cos\theta)$.

Soln The given curve

$$r = a(1+\cos\theta) \quad \text{---(1)}$$

$$\Rightarrow \frac{dr}{d\theta} = -a\sin\theta = \frac{-\sin\theta \cdot r}{1+\cos\theta} \quad \text{---(1)}$$

Replacing $\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$, we have

$$-r^2\frac{d\theta}{dr} = \frac{r\sin\theta}{1+\cos\theta}$$

$$\Rightarrow r\frac{d\theta}{dr} = \frac{\sin\theta}{1+\cos\theta}$$

$$\Rightarrow \frac{1+\cos\theta}{\sin\theta}\,d\theta = \frac{dr}{r}$$

$$\Rightarrow \frac{dr}{r} = \frac{2\cos^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\,d\theta$$

$$\Rightarrow \frac{dr}{r} = \cot\frac{\theta}{2}\,d\theta$$

Integrating, we get

$$\Rightarrow \log r = 2\log\sin\frac{\theta}{2} + \log k$$

$$= \log\sin^2\frac{\theta}{2} + \log k$$

$$= \log\frac{1-\cos\theta}{2} + \log k$$

$$\Rightarrow \log r = \log(1-\cos\theta) + \log c \quad \text{where } c = \frac{k}{2}$$

$$\Rightarrow r = c(1-\cos\theta)$$

This is the required orthogonal trajectory.

— x —

2. Find the orthogonal trajectory of the curve $r^n = a^n \cos n\theta$.

Soln.

The given curve

$$r^n = a^n \cos n\theta$$

Taking logarithm $\Rightarrow$

$$n \log r = \log (a^n \cos n\theta)$$

$$\Rightarrow n \log r = n \log a + \log \cos n\theta$$

Differentiating w.r.to θ, we get

$$\Rightarrow n \frac{1}{r} \cdot \frac{dr}{d\theta} = 0 + \frac{1}{\cos n\theta} \cdot (-\sin n\theta) \cdot n$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = -\tan n\theta$$

Replacing $\frac{dr}{d\theta}$ by $-r^2 \frac{d\theta}{dr}$, we get

$$\therefore \frac{1}{r} \times -r^2 \frac{d\theta}{dr} = -\tan n\theta$$

$$\Rightarrow r \frac{d\theta}{dr} = \tan n\theta$$

$$\Rightarrow \frac{dr}{r} = \cot n\theta \, d\theta$$

Integrating, we get

$$\log r = \frac{1}{n} \log \sin n\theta + \log k$$

$$\Rightarrow n \log r = \log \sin n\theta + n \log k$$

$$\Rightarrow \log r^n = \log \sin n\theta + \log k^n$$

$$\Rightarrow \log r^n = \log (k^n \cdot \sin n\theta)$$

$$\Rightarrow r^n = k^n \sin n\theta .$$

This is the required orthogonal trajectory.
